A Validation Process for Complex Knowledge: The Standards for Mathematical Practice Knowledge Assessment

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Abstract

The Standards for Mathematical Practice (SMPs; CCSSI, 2010) describe mathematical behaviors and habits that students should express during mathematics instruction. Teachers should have knowledge about the SMPs, their meanings, and their implications for what students are expected to do when engaging in and with mathematics. In this chapter we describe the process and development of the Standards for Mathematical Practice Knowledge Assessment (SMP-KA). The purpose of this manuscript is to share validity evidence for interpretations of outcomes from the SMP-KA as a measure of teachers' knowledge related to the SMPs. The development of the SMP-KA drew upon on Anderson and Krathwohl's (2001) framework describing Bloom's revised taxonomy. The revised taxonomy's dual dimensions of knowledge and cognitive process guided construction of the instrument to ensure that important elements of the complexity and depth of the SMPs are assessed. Knowledge of the SMPs involves both mathematics content and a knowledge of students' behaviors as they engage with mathematics. The SMPs are an important part of a mathematics teacher's knowledge for teaching. The SMP-KA is an instrument to measure inservice K-12 teachers' complex knowledge on the Standards for Mathematical Practice. The gathering of validity evidence and the instrument development process for the SMP-KA followed the standards of the American Educational Research Association, American Psychological Association, and the National Council on Measurement in Education (AERA, APA, & NCME, 2014). The chapter provides sufficient validity evidence to support use of the SMP-KA to assess teacher knowledge of the SMPs.

Knowledge about teaching is complex on many levels (Association of Mathematics Teacher Educators [AMTE], 2017; Leonard, Brooks, Barnes-Johnson, & Berry, 2010; Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013). When considering classroom teaching practice and the use of knowledge during instruction to make pedagogical decisions for students' learning, the dynamics are multifaceted (National Council of Teachers of Mathematics [NCTM], 2014). The practice of teaching mathematics involves multiple knowledge sets: knowledge of mathematics, knowledge of students, knowledge of pedagogy, knowledge of mathematical practices, knowledge of curriculum, and knowledge of social contexts (AMTE, 2017) and knowledge of mathematical and pedagogical connectedness (Matney, 2014). Because teaching draws on many knowledge types, instruments are needed to adequately measure these diverse forms of knowledge. For instance, measurement of content knowledge, pedagogical content knowledge, and classroom practices are three viable directions for measuring teachers' knowledge related to instruction. The purpose of this chapter is to share the validitation process and evidence for the Standards for Mathematical Practice Knowledge Assessment (SMP-KA), which measures teachers' knowledge about the Standards for Mathematical Practice (SMP), using a lens of depth-of-knowledge and knowledge of standards.

Purpose Statement for Standards for Mathematical Practice Knowledge Assessment

The SMP-KA was designed to measure a teacher's knowledge of the Standards for Mathematical Practice (SMP). The SMPs are delineated in the Common Core State Standards for Mathematics (CCSM) as eight standards elucidating what it means to be mathematically proficient (Chief Council of State School Officers [CCSSI], 2010). The SMP-KA uses the Revised Bloom's Taxonomy (Anderson & Krathwohl, 2001) as a framework. A major portion of the United States' 3.1 million teachers (Common Core of Data, 2018) is in the 42 states that *expect teachers to know and use the SMPs* as part of their professional teaching obligation. The SMP-KA has four phases (see Appendix A for a concise explanation of the phases) and takes between 30 minutes to 60 minutes to administer. The target population for the SMP-KA is K-12 teachers in states that adopted the CCSSM. While the SMP-KA was developed for use with both preservice and inservice teachers, this chapter only presents and analyzes the validity evidence pertaining to its use with inservice teachers. Measurement contexts include, but are not limited, to pre- and post-testing for professional development evaluation and/or research on teacher's SMP-related knowledge. It is strongly recommended for use when considering conditions under which it might be conjectured that teachers are developing knowledge of the SMPs in robust ways, either through long-term professional development or district initiatives involving mathematical proficiency. The assessment is given via computer in which participants type their responses in a single sitting. Participants complete the four phases of the assessment one at a time and, upon completing each phase, they are locked out of going back to a prior phase. The items are not released to the public to protect the integrity of the SMP-KA. Interested users should contact the first author for pricing and use information. Only trained reviewers, who have completed a minimum of 100 hours of SMP-related professional development, score responses. Due to the extensive professional knowledge required for scoring, end-users must contract with the developers for reliable scoring. The results of the scores are then returned to the end-user for analysis. A total raw score may be summed across all phases of the SMP-KA for a total of 81 points (e.g. 47/81).

Theoretical Framework of Knowledge

Forty-two states in the USA have adopted the CCSSM (CCSSI, 2010) in some form as of 2018. The CCSSM include the Standards for Mathematical Practice (SMPs) and the Standards for Mathematics Content. The eight SMPs (see Table 1) describe mathematical behaviors and habits that students should demonstrate and teachers should seek to encourage through planning and implementation (Bostic, Matney, & Sondergeld, in press). Because these are standards that teachers should promote during classroom instruction, it seems reasonable to infer that teachers should know what they are. Unfortunately, the rollout of new standards included insufficient funding for professional development and inadequate time to support teachers' growth as professionals (Bostic & Matney, 2013). Fast-forward to 2018 and teachers continue to experience professional development focused on the SMPs and concomitantly, preservice teachers are learning about them through university coursework and field experiences (Kruse, Schlosser, & Bostic, 2017).

Insert Table 1 here

Few measures are designed to reliably capture teachers knowledge of the SMPs. The Mathematics Classroom Observation Protocol for Practices (MCOP^2; Gleason, Livers, & Zelkowski, 2017) and the Revised Standards for Mathematical Practice Look-for Protocol (Bostic et al., in press) allow users to connect observational data with the SMPs. While both measures are grounded effectively in robust validation evidence, there are no measures of a teachers' knowledge of the SMPs. Without such a measure, it is unclear whether there is a viable means to connect a teachers' knowledge with classroom practices. Thus, there is a need to develop and validate a measure so that teachers' knowledge and practice can be more adequately connected and investigated.

The SMPs (CCSSI, 2010) themselves are complex notions of what it means to engage with and construct knowledge of mathematics. In the development of the Standards for Mathematical Practice Knowledge Assessment (SMP-KA), we sought a framework that would treat complex knowledge (creating and analyzing ideas) as being different from basic knowledge (e.g., recalling ideas) and that would guide the construction of items to focus on different kinds of teacher knowledge about the SMPs. The revision of Bloom's Taxonomy (Anderson & Krathwohl, 2001) was chosen as an orienting framework for the SMP-KA due to the taxonomy's usefulness in distinguishing different types of knowledge with dual dimensionality. The taxonomy has two distinctive dimensions: *Cognitive Process and Knowledge*. The Cognitive Process dimension has six categories: Remember, Understand, Apply, Analyze, Evaluate, and Create. The Knowledge dimension has four categories: Factual, Conceptual, Procedural, and Metacognitive. These categories are thought to vary in complexity and as such form a hierarchy (Krawthwohl, 2002). Taken collectively, these two dimensions form a matrix of 24 possibilities for classifying knowledge. For instance, knowledge might be considered to fit within the Analyze and Conceptual cell of the matrix, and could be described as "analyze conceptual knowledge" (ACK).

The SMP-KA does not measure all knowledge a teacher might have about the SMPs. The limitations of a survey inhibit what can be revealed about actual instruction, such as how teachers' use their knowledge of SMPs to plan and enact instruction. The field has done well to develop a few instruments with validity evidence that measure teachers' use of SMPs within observable instruction (see Bostic et al., in press; Gleason et al., 2017). These instruments allow researchers to gather evidence of the execution and implementation of knowledge, both of which are associated with the cognitive dimension of application in the Revised Taxonomy (Anderson & Krathwohl, 2001). What the field lacks however, is a critical examination of the other knowledge categories such as factual knowledge and conceptual knowledge; the SMP-KA fills much of this needed gap. This chapter provides a validity argument for outcome interpretations from the SMP-KA's assessment of teachers' knowledge in the areas of Remembering Factual Knowledge (RFK), Understanding Conceptual Knowledge (UCK), Analyzing Conceptual Knowledge (ACK), Creating Factual Knowledge (CFK), and Creating Conceptual Knowledge (CCK).

Instrument Validation Process

We drew upon our own previous work and the work of others that addresses validating outcomes from measures and assessments (American Educational Research Association, American Psychological Association, National Council on Measurement in Education [AERA, APA, & NCME], 2014; Bostic et al, in press; Bostic, Sondergeld, Folger, & Kruse, 2017). Table 2 lists the action steps as well as their connections to validity. Following table 2 are detailed descriptions of the procedures and analysis for validating outcomes.

Insert table 2 here.

Stage One

In considering literature related to the SMPs, we began by analyzing the SMP paragraphs from the Common Core State Standards for Mathematics (CCSSI, 2010, see pp. 6-8). We read each SMP four times using different lenses. Upon the first read through, we looked for the keywords that illuminated the big picture of the SMP. The second read through focused on separating mathematical examples from more general statements about the SMP. On the third read through, we considered how the examples for each SMP connected to the general statements. This was done for the purpose of considering future examples that would honor the intent of the SMP. The fourth read through was done to find explicit connections across all eight SMPs.

After giving focused attention to understanding the SMPs, an examination of literature written by education experts since 2010 occurred via a review of manuscripts, proceedings, and presentations about SMPs. In addition to research literature, other expert ideas about the SMPs were sought from nationally known groups that have profound influence in mathematics education (e.g., NCTM and National Council of Supervisors of Mathematics). This literature included books such as *Connecting the NCTM Process Standards and the CCSSM Practices* (Koestler, Felton, Bieda, Otten, 2013) and *Principles to Action: Ensuring Mathematical Success for All* (NCTM, 2014). The literature provided insights about the significance of the ideas found in the SMPs, specifically what knowledge teachers need to promote the SMPs and why it is important for teachers to have knowledge of the SMPs. While the action steps completed at this stage did not provide validity evidence to support the intended use of the SMP-KA, it was an important first step in understanding the depth and breadth of the knowledge of the SMPs that the instrument was being designed to measure, allowing developers to collect appropriate validity evidence based on test content during stage five (AERA et al., 2014).

Stage Two

An expert panel consisting of 31 individuals was created to ascertain data about knowledge of the SMPs. All members of the panel were selected because of their extensive work involving the SMPs (nature of the involvement described below) and their intended promotion of the SMPs through their respective positions. The representatively selected expert panel included mathematics teaching professionals from nine different perspectives: K-5, 6-8, and 9-12 mathematics teachers, mathematics instructional coaches, mathematicians, mathematics education graduate students, mathematics educators, a mathematics curriculum coordinator, and a state department of education mathematics representative.

K-12 mathematics teachers participating in the expert panel had completed more than 120 hours of SMP focused professional development. Each teacher had worked to promote student engagement in the SMPs and previously, had shown instruction that met or exceeded norms on the Revised Standards for Mathematical Practice Look-for Protocol (Bostic et al., in press; Bostic & Matney, 2014). The teachers' professional experience ranged from five to 28 years. There were six teachers from grades K-5, six teachers from grades 6-8, and six teachers from grades 9-12, which resulted in a total of 18 teachers. There was at least one teacher from each grade-level on the expert panel. We sought input from two mathematics coaches, a curriculum coordinator, and a state department of education representative. These experts work at the district- or state-level to support mathematics teachers in the promotion of the SMPs during instruction. We reached out to obtain feedback from two mathematicians who held a terminal degree (Ph.D). The mathematicians were part of professional development teams who worked with K-12 mathematics teachers on promoting the SMPs during instruction. They have taught a variety of graduate mathematics courses and courses for K-12 mathematics teachers and preservice teachers. Four mathematics education graduate students provided input based on their work to create, enact, and research grant-funded professional development about SMP instruction. Lastly, three mathematics teacher educators holding Ph.D.s, and hailing from different states, were asked to serve on the panel. These experts were selected due to their research involving SMPs, which had been presented at peer-refereed national mathematics education conferences and/or published in peer-reviewed mathematics education journals. During this stage members of the expert panel communicated with instrument developers in face-to-face meetings, telephone calls, and email. Each communication was done individually and no expert panel member had knowledge or influence on the others' responses. For developers to fully understand the knowledge content that must be measured by the SMP-KA, developers asked the panel to respond to the following questions:

- What knowledge of the SMPs are useful in your own work?
- How much do teachers you work with know about the SMPs? What kinds of things do they know? The titles, examples, other things?
- What should teachers know about the SMPs in order to be effective at promoting the SMPs through their teaching with the students?

The panel provided important data about which SMPs teachers may find difficult to understand. For example, nearly all of the panel members described both their own initial difficulty, as well as colleagues' difficultlies, in making sense of how SMP7 and SMP8 are alike and different. For many teachers, there was a subtle nuanced difference between students noticing a pattern or structure in SMP7 and students recognizing repeated reasoning in SMP8.

The panel also shared examples of misconceptions that teachers might have. The most common misconception discussed by the panel was that some teachers confuse the strategic use of manipulatives by students to solve a problem (SMP5) as necessarily meaning that students were modeling with mathematics (SMP4). As the word “model” holds many meanings in the English language, such confusions are perhaps quite natural at first and suggest that teachers need opportunities to become more knowledgeable about the meaning of the SMPs. Other important data the panel provided involved the connections between the SMPs themselves. Teachers who know about the SMPs will see connections between them, such as pausing in the process to consider the contextual meaning (SMP2) and the behavior of students who “routinely interpret mathematical results in the context of the situation and reflect on whether the results make sense” (SMP4; CCSSI, 2010, p. 7). Teachers knowledgeable about the SMPs might then further note that this kind of contextual mathematical thinking is similar to students being able to “maintain oversight of the process, while attending to the details” (SMP8; CCSSI, 2010, p. 8). The ideas presented by the expert panel were analyzed in stage three to develop item types for an assessment that would capture data about teachers’ knowledge of the SMPs. In addition, similar to stage one, the action steps completed during stage two provided a different perspective on knowledge of the SMPs for collection of validity evidence based on test content at a later stage.

Stage Three

In this stage, typological analysis was used to systematically analyze data drawn from the literature review and expert panel (Hatch, 2002). In typological analysis, data are divided into categories based on a predetermined framework (LeCompte & Preissle, 1993). For the SMP-KA, the Revised Taxonomy (Andersong & Krathwohl, 2001) was used to consider what types of items should be included in the instrument to ensure appropriate measurement of the different kinds of Knowledge and Cognitive Process dimensions involving the SMPs. The research team considered all 24 cross-dimensional possibilities of the Revised Taxonomy.

First, the data were read and memos were made connecting the evidence from the literature and expert panel's statements concerning the types of important knowledge for SMPs. In the second step, the main ideas in each of the identified knowledge types were recorded. Step three was to reconsider the data with a focus on each of the identified knowledge types to ensure that all codes associated with each knowledge type were documented.

In step four, the research team sought relationships within each typology. The aim of this step was to consider which knowledge types had viable evidence from the literature and /or ideas from the expert panel warranting their inclusion in the SMP-KA. For example, in order for the knowledge type, Analyzing Conceptual Knowledge, to be viable within the SMP-KA, there must be evidence from the data that it is important for teachers to be able to break down the SMPs, detecting how they relate to one another having knowledge of "the interrelationships among the basic elements" found within the "larger structure that enables them to function together" (Krathwohl, 2002, p. 214). From such analyses, five viable knowledge types emerged: Remembering Factual Knowledge (RFK), Understanding Conceptual Knowledge (UCK), Analyzing Conceptual Knowledge (ACK), Creating Factual Knowledge (CFK), and Creating Conceptual Knowledge (CCK). In step five, the literature and ideas from the expert panel were investigated again for evidence that would counter these five knowledge types. Although no counter evidence was found, we noted that one mathematics coach mentioned, "When an expert teacher knows how to really engage students in becoming mathematical thinkers, I wonder whether or not they can do that, even if they have no idea what the SMP titles are? It seems like they could. So, while knowing the titles could be beneficial I'm not sure it's necessary." The idea here that teachers could know about habits of effective mathematical thinking without knowing the factual titles of the SMPs is important. The mathematics coach is pointing out that a teacher might be able to demonstrate that they understand the SMPs conceptually (UCK) even if they cannot recite the titles of the SMPs (RFK). Thus, the research team was alerted to carefully consider the formation of items and phases (stage 4) that allowed teachers to show their knowledge of RFK independently of other knowledge types and vice-versa.

In the final step, we selected data excerpts that supported the emergence of the five knowledge types. These data excerpts were used to guide item construction for each of the five knowledge types. See Appendix B for exemplary evidence related to the five knowledge types. Similar to stages one and two described above, stage three did not provide validity evidence to support the proposed use of the SMP-KA, per se. Instead, this stage provided developers with a thorough understanding of how the Revised Taxonomy (Andersong & Krathwohl, 2001) applies to knowledge of the SMPs. Thus, when validity evidence related to internal structure (as described in AERA et al., 2014) was collected in stages seven and eight, it could be compared to the theory which informed development of the SMP-KA.

Stage Four

Drawing upon results of the typological analysis in stage three, items were created for each of the five knowledge types. For RFK and UCK items, lists of factual and conceptual knowledge were fashioned from the SMP titles, the literature, and the SMP paragraphs. These lists consisted of phrases or sentences describing a focused aspect of the SMP. For example, two aspects associated with SMP1 would be that students look for entry points to a problem's solution and students do not give up after the first attempt. Next, each aspect was analyzed by cross referencing the aspect to knowledge type, RFK or UCK. These aspects formed the basis for items about teachers' factual knowledge (RFK) and conceptual understanding of the SMPs (UCK).

For the development of items associated with ACK, CFK, and CCK, written exemplary scenarios of students engaging in the SMPs while solving mathematics problems were developed. The scenarios were created from actual events with K-12 students who were solving mathematics problems. They were taken from both live observations and videos (n = 591) of teaching where the teacher's lesson plan had an explicit focus on the promotion of at least one SMP. The videos of classroom instruction came from K-12 teachers participating in a three-year professional development program focused on SMPs. The purpose of developing these scenarios was to draw upon genuine classroom happenings to capture students' engagement in the SMPs while doing specific mathematics problems. Both the process and the product of writing SMP focused scenarios formed the basis for items about analyzing SMPs (ACK) and creating scenarios of SMPs that demonstrate factual and conceptual knowledge (CFK & CCK).

After the construction of the items, the assessment was organized into four phases (see Appendix A). The phases were then ordered in a way that would not reveal ideas about the SMPs before the teachers completed each phase. Every participant completed phase 1, phase 2, phase 3, and finally phase 4, in that order and once a phase was completed teachers could not go back to a previous phase. For example, the ordering does not allow teachers to see an exemplar scenario and then analyze it before they are asked to create one of their own, because doing so might attune teachers to what is possible and obfuscate assessment of their prior knowledge of the SMPs. Table 3 shows the alignment and order of each section of the SMP-KA with its associated knowledge type. Three of the phases (1, 2, and 4) are each associated with exactly one knowledge type. Phase 3 is associated with two knowledge types. Krathwohl (2002) explains that particular items and objects will sometimes require participants to engage in more than one type of knowledge. In the case of Phase 3, the nature of the task requires participants to create fractual knowledge and create conceptual knowledge (see appendix A for more detail). Developing items that indicate what a teacher knows about the SMPs requires thoughtful consideration. Close attention was paid to what was written in the SMPs (CCSSI, 2010) as well as the ways the mathematics education literature and experts described the SMPs. Furthermore, we looked at actual classroom happenings, via the videos mentioned previously, to ensure the items fit within the contexts of student's mathematical problem solving.

Insert Table 3 here

Stage Five

A goal of this stage was to collecte validity evidence based on test content. Such evidence connects the content within an instrument and the construct it intends to measure (AERA et al., 2014) and in turn, provides assurances that the score interpretations are appropriately drawn. To collect validity evidence based on test content (AERA et al., 2014), an expert panel of 13 individuals reviewed the assessment developed from the previous stage. It consisted of a representative sample of individuals from each group described earlier in stage two; two terminally degreed mathematicians, two terminally degreed mathematics educators, three teachers from each grade band (K-5, 6-8, 9-12), three mathematics education graduate assistants, two mathematics coaches from elementary (K-8) and secondary (7-12), and one mathematics curriculum coordinator. The panel examined the SMP-KA and gave verbal or written feedback about the connections they made between the items and knowledge of the SMPs. Panel members were asked to directly respond to the appropriateness of the items in representing a teacher's knowledge about the SMPs. This was done for each knowledge type (RFK, UCK, CFK, CCK, and ACK). The feedback was positive and suggestions for improvement were shared. The assessment was modified to include connections between the SMPs where they were warranted and some of the exemplar scenarios were eliminated based on the experts' feedback. Including all of the scenarios made the assessment very long. The consensus of the panel was that one exemplar scenario per SMP was sufficient. Eight scenarios were selected that the panel agreed were representative of the eight SMPs. In other words, for each SMP there was exactly one matching scenario. These modifications were made to the instrument before collecting data during stage six.

Stage Six

Cognitive interviews can be used to improve the validity and reliability of research tools (Desimone & Le Floch, 2004) and to provide validity evidence based on response processes (AERA et al., 2014; Castillo-Díaz & Padilla, 2013). One-on-one cognitive interviews were conducted with participants who would be expected to take the assessment (preservice and inservice teachers) and potential users of the assessment results. There were a total of 15 participants during this stage; six preservice teachers, six inservice teachers, one mathematics coach, one mathematics curriculum coordinator, and one terminally degreed mathematics educator. The focus of the cognitive interviews was to inquire about the meaning of the items by user and to gauge the assessments ease of use via the computer. The cognitive interviews were transcribed and inductive analysis (Hatch, 2002) was conducted to determine any themes or discrepancies in meanings of items across potential assessment takers and assessment users. The results of our analyses revealed that all phases of the assessment were interpreted as intended and easy to use via the computer; except one. These interviews rewarded us with important information about how the directions for phase two were being interpreted. The problematic interpretation of the directions did not elicit the desired information about teachers' knowledge of the SMPs. The directions for phase two were reworded to be more direct and clear for participants. After rewording the directions, the teachers gave feedback about the expectations of the directions that aligned with the intent. This change was made prior to stage seven.

Stage Seven

In this stage, we piloted the assessment with K-12 teachers from a Midwest state that adopted the CCSSM in 2011, five full years before teacher participants joined this validation study. Teachers who participated in the pilot were enrolled in a grant that included professional development (PD), which had the study of the SMPs as one of its stated goals. The PD included more than 68 hours of professional development that either directly examined the descriptions of the SMPs (CCSSI, 2010) or explored the SMPs through the context of problem solving and student engagement. Teachers took the assessment on two occasions. The first administration was prior to any grant-related professional development and then again after the conclusion of the grant. The teachers came from rural, suburban, and urban schools. Most of the teachers were female and Caucasian. During this stage, 189 participants completed the SMP-KA 285 times, with half of the participants (n = 94/189, 49.7%) completing the assessment both before and after the grant activities. The pilot administration of the SMP-KA at this stage provided the data that was analyzed during the next stage to provide validity evidence based on internal structure (AERA et al., 2014) and evidence of the instrument's reliability and internal consistency.

Stage Eight

We performed reliability analyses and exploratory factor analyses to determine the psychometric properties and performance of the SMP-KA. As a first step to determine instrument reliability, interrater agreement was calculated for five raters. Due to the complexity of SMP knowledge, only raters who had more than 100 hours of SMP professional development were allowed. Each rater underwent about ten additional hours of training. The training process was done in teams of two or more and began with two hours of SMP-KA Rubric reading followed by making explicit connections between the rubric, the SMP paragraphs (CCSSI, 2010) and literature. Next, the raters were given data from five participants. These data were chosen to show the breadth, depth, and variance of teachers' answers. The raters then scored the first participants' data together with the first author allowing for discussion on the meanings of the participants' responses and how to code consistently. Next, the raters scored four more participants' data independently. After independent scoring, the raters reconvened with the first author to discuss each score. Any discrepancies between the scores were discussed until agreement was made. After this training and calibration experience, each rater scored six new assessments independently. These independent assessments were then analyzed and revealed 97.1% exact agreement across coders. This exceeds the minimum threshold of 90% needed to conduct reliability analyses (James, Demaree, & Wolf, 1993), and represents a stronger indicator of consistency than interrater reliability insofar as it suggests that raters interpret participant responses similarly and assign codes virtually identically (Gall et al., 2007).

Exploratory factor analyses (EFA) were used to understand the underlying structure of the instrument. EFA was appropriate in this case, not only because these items were just developed and had not been analyzed together previously (Bandalos & Finney, 2010), but also because of the theory on which they were designed. The instrument was developed to conform to two different theoretical structures simultaneously; each phase of the assessment measured different dimensions of knowledge according to the Revised Taxonomy (Anderson & Krathwohl, 2001), and each phase also contained items about each of the SMPs (CCSSI, 2010). As such, it was reasonable to use EFA rather than confirmatory factor analysis (CFA), as it is unclear to which theoretical framework the emergent sub-factors would align (Bandalos, & Finney, 2010). The SMP-KA measures teachers' deep and complex knowledge of the SMPs, which was conceptualized as a single higher-order factor that incorporated all of the information contained in the sub-factors that might emerge. Results of the EFA offered insight into the psychometric qualities of the SMP-KA and allowed researchers to interpret subscale scores to examine distinct, though related, aspects of the knowledge it measures. We conducted EFA using maximum likelihood estimation in SPSS Statistics, version 24, (IBM Corp., 2016). Because factors were expected to correlate, we selected an oblique rotated solution using the Promax method with Kaiser normalization (Field, 2018). Four factors emerged with eigenvalues greater than one. Inspection of the scree plot (see Figure 1) revealed an inflection point at four factors, confirming the four-factor solution (Field, 2018).

Insert Figure 1 here

Each identified factor contained the items included in each phase of the assessment and jointly explained 67.0% of the total variance in participant scores before rotation. After rotation, factors overlap preventing calculation of total variance explained (Bandalos, & Finney, 2010), but the sums of squared structure coefficients for each factor can be used to compare the amount of variance uniquely explained by each factor. Table 4 displays the means, standard deviations, pattern coefficients (which represent the unique relationship between the item and the underlying factor after controlling for the other factors), and communalities for each item in the SMP-KA, as well as the phase, factor, sum of squared structure coefficients, and Cronbach's alpha value (discussed next) for each grouping of items.

Insert Table 4 here

Analysis of the internal consistency for each subscale using Cronbach's alpha indicates reliabilities either above .90 or, in the case of the CFK/CCK factor, very near it, and are suitable for the purposes of this instrument (Lance, Butts, & Michels, 2006). The overall scale that included all items in the instrument demonstrates high reliability with an internal consistency of $\alpha = .97$, which is excellent (Lance et al., 2006). Internal consistency analyses for the overall scale, and for each subscale, indicated that dropping any of the items associated with that scale would not increase the internal consistency, suggesting that each item contributes well to the measure of both teachers' complex content knowledge of the SMPs and of the its corresponding knowledge dimension. Factor correlations (see Table 5) suggest that the use of oblique rotation was appropriate, as all four factors are highly correlated (Cohen, 1992). These high correlations also suggest that, although the scores for each phase can provide information on its associated knowledge dimension, the results of the SMP-KA are best interpreted as a single total score representing a teacher's complex knowledge of the SMPs.

Insert Table 5 here

Discussion

The SMP-KA was carefully designed to address a gap in the measurement of teachers' knowledge of the SPMs. Using the Revised Taxonomy (Anderson & Krathwohl, 2001) as a framework, the SMP-KA accounts for measures of teacher's knowledge within five domains (RFK, UCK, ACK, CFK, and CCK) that were previously unmeasured. Furthermore, the SMP-KA gives a total score by which an increase of these knowledge domains can be measured as a teacher's knowledge of the SMPs expands. These data may interest schools and districts who want to know about teachers' knowledge of the SMPs, in order to make data-based decisions on future professional development, analyze district compliance with implementation of the standards, measure the level of preservice or inservice teacher knowledge of SMPs, or consider how teacher knowledge connects with their students' mathematical behaviors and actions. Similarly, researchers wanting to investigate teachers' knowledge of the SMP's could employ the SMP-KA in order to find out the effects of SMP related professional development or other correlated variables.

The development of the SMP-KA was done to complement the observational protocols related to the SMPs (see Bostic et al., in press; Gleason et al., 2017). These observation protocols may include data for assessing teacher's knowledge of specific mathematics skills and algorithms, implementation of techniques and pedagogical procedures related to promotion of the SMPs, and knowledge of when to apply appropriate mathematical and pedagogical procedures. Classroom observation data have the power to reveal teacher's execution and implementation of the SMPs, and hence are particularly well suited to give robust information about teacher's SMP knowledge in the domains of application of factual knowledge, application of conceptual knowledge, and more broadly in teachers' remembering, understanding, applying, analyzing, and evaluating procedural knowledge of the SMPs. On the other hand, these instruments do not give consideration to other areas of knowledge, and it was within these spaces that our development of the SMP-KA sought to inquire about other areas of knowledge measurement would be valuable to the field. The expert panel revealed five untapped areas of knowledge (RFK, UCK, ACK, CFK, CCK) for the SMP-KA to assess. It seems plausible then, that those looking to have a large swath of information related to teachers' knowledge of the SMPs could couple the SMP-KA with classroom observation protocols, in order to gain a strong picture of teachers' SMP knowledge across several domains.

Participants in this study worked through the items on the SMP-KA at different rates, ranging from 30 minutes to 60 minutes. Based on these analyses and on the idea that the eight SMPs constitute a single, overarching psychological construct that includes a variety of diverse mathematical behaviors and habits, it is plausible that a short form of the SMP-KA can be developed, which would provide similar measurement quality while requiring less time to complete the assessment. Such a short form must be systematically constructed with careful attention to the EFA and internal consistency analyses and to align with theory.

Future Research and Limitations

As mentioned in the discussion section the SMP-KA is a measure of teachers' SMP knowledge and is not a measure of enacted instruction. Teaching mathematics is a complex endeavor (AMTE, 2017) in which teacher's use an array of knowledge in their planning and in the moment by moment orchestration of the student learning. Although knowledge plays an important role in all elements of teaching, consideration of a teacher's knowledge, by itself, limits teachers, school leaders, researchers, and policy makers' abilities to make decisions about what is needed to improve teaching and learning. Thus, future research should explore evidence connecting knowledge of the SMPs with enacted SMP instruction. Similarly, the SMP-KA may or may not show correlation to student outcome measures, including but not limited to, achievement, problem solving, and affect.

The field of mathematics education has long developed knowledge measures for teachers in various knowledge domains, such as the DTAMS (Saderholm, Ronau, Brown, & Collins, 2010) and LMT (Ball, Thames, Phelps, 2008) for mathematical knowledge for teaching. Researchers have also examined under what professional development conditions these measures tend to perform better (Copur-Gencturk & Lubienski, 2013). The SMP-KA provides a direct and focused measure of teachers' knowledge of the SMPs. Future research should explore how teachers' knowledge of the SMPs connect with other knowledge measures that are frequently used in the field as well as the professional development conditions under which teachers' knowledge of SMPs provides substantive benefits.

At this time, all of the work done with the SMP-KA has been by researchers and mathematics education graduate students who were part of the development team. Furthermore, the participants for the validity examination came from professional development programs created by the developers in the years prior to its development. For these reasons, further research is needed to see what happens when others use the SMP-KA.

Final Thoughts

This chapter provides validity evidence for the score interpretations from the SMP-KA as a measure of inservice teacher's complex knowledge of the SMPs, and connects that evidence to the measure development process. Table 6 concisely shows the sources of evidence, the validity claims, and supporting evidence given in this chapter.

Insert Table 6 here

As shared in table 6, this is not the only time that validity evidence will be gathered because validation is an ongoing process (AERA et al., 2014; Bostic, Krupa, Carney, & Shih, in press; Kane, 2012). For example, validity evidence data need to be gathered from preservice teachers to explore how their score interpretations are similar or different to inservice teachers. Another opportunity for futher research involves gathering validity evidence in relation to other variables by exploring connections between measures of pedagogical content knowledge (e.g., DTAMS and LMT) and the SMP-KA. We will continue to explore ways to strengthen the validity argument for this measure of teachers' knowledge of the SMPs.

The SMP-KA provides a means to measure teachers' knowledge of the SMPs. As such, schools, districts, or states in which the CCSSM was adopted may use the SMP-KA as a gage for where teachers are in their knowledge of mathematical proficiency. Those providing professional development involving learning about SMPs, or what mathematical proficiency means in the CCSSM, might use the SMP-KA to gather evidence of changes in teacher knowledge. Researchers who are interested in the relationship between teachers' knowledge and instruction may use the SMP-KA as one tool in studies exploring the enacted dynamics of teachers' knowledge about mathematical proficiency. With that said, the SMP-KA provides scholars with a means to explore teachers' knowledge of the SMPs connected to numerous contexts. There is sufficient validty evidence related to a variety of sources necessary to ground the score interpretations, when the SMP-KA is used appropriately. And in sum, this chapter may help readers who are thinking about how the measure development process and validity gathering process occur in tandum with one another.

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Appendix A: Phases and Dual Knowledge Dimensions

The SMP-KA has four phases of knowledge assessment developed to engage the responder in five different dual dimensions of knowledge and cognitive processes (Andersong & Krathwohl, 2001).

Phase 1 – Remembering factual knowledge of the SMP's (RFK)
Phase 2 – Understanding conceptual qualities of the SMP's (UCK)
Phase 3 – Creating scenarios of student engagement with factual Knowledge *and* conceptual knowledge of the SMP's (CFK/CCK)
Phase 4 – Analyzing scenarios of student engagement in the SMP's for conceptual knowledge (ACK)

Examples of Phases

Phase 1 – Remembering Factual Knowledge
In this phase the assessment asks the respondents to list the titles of the eight practices. The list does not have to be word for word or in the common order found in the CCSSM (CCSSI, 2010).

> Example Response (**SMP1**): "Students should makes sense of each math problem they do and never give up trying to solve it."

Phase 2 – Understanding Conceptual Knowledge
In this phase the respondents are given the eight SMP titles and asked to list qualities exemplifying each standard that are representative of mathematically proficient student behaviors for that standard.

> Example Response (**SMP1**): "Students try to figure out for themselves the meaning of a problem. Students look for entry points to a problems solution. Students analyze givens, constraints, relationships, and goals."

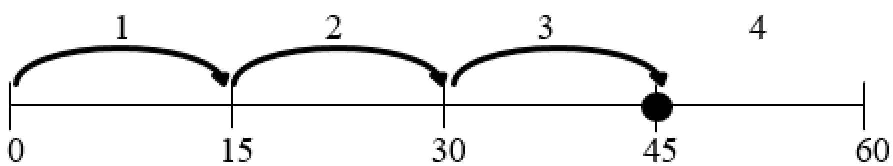
Phase 3 – Creating Factual and Conceptual Knowledge
In this phase the assessment asks the respondents to give an example scenario of when a students is exhibiting engagement in the SMP. The respondents are instructed to describe the mathematical situation (problem) and explain how the student is exhibiting each SMP.

> Example Response for Decontextualizing in **SMP 2**: "The students were given the following problem: "John and Mark brought oranges to school to share with the class. John was not sure how many he had but when Mark added his 16 oranges to John's every student in the class had exactly 1 orange of their own. If there are 35 students in the class how many oranges did John bring?" In considering this problem the students notice that they can make this problem into the equation $X + 16 = 35$. The students use the properties of algebra to solve for X noting they can take equal amounts away from both sides (take away 16) to get an equivalent equation revealing what John's amount should be. $X = 19$."

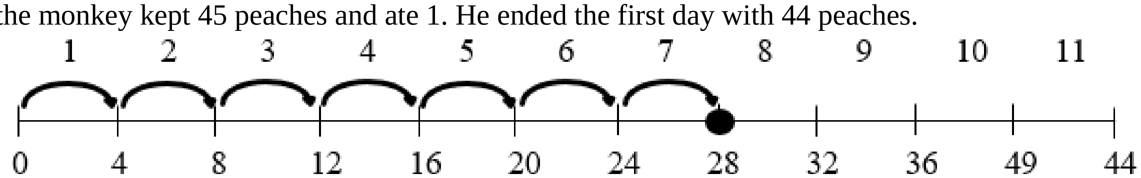
Phase 4 – Analyzing Conceptual Knowledge
In this phase respondents are given a scenario in which students are engaging in mathematics problem solving. The scenario may have multiple SMP references but it has an over-arching theme throughout. Respondents are asked to identify which SMP is themed by the scenario.

The following example is one of the pilot scenarios that was not chosen for the instrument. In order to preserve the integrity of the SMP-KA the scenarios in the instrument are not made public.
Example Scenario for **SMP 5**: Peaches Today…Peaches Tomorrow…
Students were given the problem: "A little monkey had 60 peaches. On the first day he decided to keep $\frac{3}{4}$ of his peaches. He gave the rest away. Then he ate one. On the second day he decided to keep $\frac{7}{11}$ of

his peaches. He gave the rest away. Then he ate one. On the third day, he decided to keep $\frac{5}{9}$ of his peaches. He gave the rest away. Then he ate one. On the fourth day, he decided to keep $\frac{2}{7}$ of his peaches. He gave the rest away. Then he ate one. On the fifth day he decided to keep $\frac{2}{3}$ of his peaches. He gave the rest away. Then he ate one. How many did he have left at the end?" Students had access to snap cubes, fraction tiles, fraction circles, and counters. One student had worked on a similar fractions problem before and used counters. She went to grab the counters and try the strategy that worked before. As she was laying out the 60 counters, she was realizing that it was taking a long time to organize that many counters. She decided that counters weren't going to be the most efficient strategy. She considered the other tools available and thought about fraction tiles or fraction circles, but the denominators in the problem do not match with the denominators on the tiles or circles. The tiles or circles do not have $\frac{1}{7}$, $\frac{1}{9}$, and $\frac{1}{11}$ unit fractions. She then decided the best method was going to be paper and pencil. She began by drawing number lines to find the amount of peaches the monkey had at the end of every day.



In order to find $\frac{3}{4}$ of 60, she first found $\frac{1}{4}$ of 60 by dividing 60 by 4 to get 15. Using her paper and pencil, she created a number line with 4 intervals of 15. She saw that 45 was $\frac{3}{4}$ of 60. On the first day, the monkey kept 45 peaches and ate 1. He ended the first day with 44 peaches.



In order to find $\frac{7}{11}$ of 44, she first found $\frac{1}{11}$ of 44 by dividing 44 by 11 to get 4. She created a number line with 11 intervals of 4. She saw that 28 was $\frac{7}{11}$ of 44. On the second day, the monkey kept 28 peaches and ate 1. The monkey ended the second day with 27 peaches. The student continued to use the number line strategy for each day to determine that monkey had 1 peach at the end of the fifth day.

Appendix B: Excerpts from Typological Analysis on the Five Knowledge Types

Knowledge Type	Expert Panel Data
Remember Factual Knowledge	"Most of my colleagues cannot even name, umm the SMPs. They have never read them and have no idea they are part of our standards for teaching math. I think if they even just knew that they would be better off because then they could think about whether or not they even asked their students to construct a viable argument or consider structure." "I, actually ashamed to say this, but I had never even read the Standards for Mathematical Practice prior to taking the PD on it and that was 3 years after state adoption. Eeek! I'm definitely not alone either. Teachers need to know about the SMPs. The titles are a great place to start..." "I think teachers need to have deep knowledge of the SMPs but most do not even know the names. When I ask them what SMP they want their students to engage in they often fumble around looking for the document that I sent them. I mean, they send me fully developed lesson plans but they haven't even considered the SMPs because they don't know the names much less what would be involved in instruction."
Understanding Conceptual Knowledge	"Teachers need to understand what it means to think about mathematics. The SMPs are about the way we engage in thinking about mathematics, the way we make sense of and create new mathematics. Understanding that is essential for teachers." "Because we have had PD I find the SMPs less intimidating. I can now, you know, classify and state what kinds of things students do when they do SMPs. So that's important for all teachers because like, what do the SMPs mean students should be doing? To know that is extremely helpful in every area, you know like, planning, teaching, and assessment." "Teachers should know what is in the SMPs; what exemplifies each one for students. I think they should be able to explain how students engage each practice. And be able to distinguish which students are engaging in it or not (comparing)."

Analyzing Conceptual Knowledge	"What happens during instruction is a lot. As we are teaching there are many things to attend to with management, the content, and the practices. To do that job we [teachers] have to be able to quickly note what kinds of things students are doing that are attributes of the SMP they want to promote. It's so essential for us to think about this before, like we do in Lesson Study, because if we do not think about it before we don't notice that it's happening when it does." "So I think teachers should really be able to connect the pedagogical elements of teaching to student's mathematical thinking and ask themselves, 'How can I encourage them to engage in this SMP?' Or like, let's say that they watch a video or see another teacher teaching, [pause] if the teacher can point out how what is happening with the teaching and what the students are doing as evidence of an SMP then that would tell me that they 'know' the SMP.
Creating Factual and Conceptual Knowledge	"Essentially teachers create. We create plans to teach an idea that we hope students want to learn about and the SMPs are the same. We want to create the plan in such a way that students are doing these SMPs. When I didn't know about the SMPs it was the farthest thing from my mind to plan for them. But now after our Lesson Studies and everything we have done I know how to plan to help students do the SMPs." "I really think it would help all teachers to make sense of the SMPs and really think through what they look like in practice. That starts with their own engagement in mathematics but it can't just stay there. They [teachers] need to understand what it means to design lessons in which students are given rich mathematics tasks that draw out ways of encountering mathematics, you know, so that the students are doing the SMP's. Knowing how to visualize SMP's through the design of lessons is a must in my opinion because if we [teachers] can do that it would improve all students' math learning." "Teachers need to know how to create learning spaces in which students are not only thinking about mathematics content, but thinking about it in such a way that honors the SMPs. I think these practices are pivotal to student learning about any content. So what I mean is, teachers who have a good knowledge of the SMPs should be able to visualize and describe the kind of happenings during instruction that exemplify the SMPs."

Counter Evidence	"So like when an expert teacher knows how to really engage students in becoming mathematical thinkers I wonder whether or not they can do that even if they have no idea what the SMP titles are. It seems like they could. So while knowing the titles could be beneficial I'm not sure it's necessary."

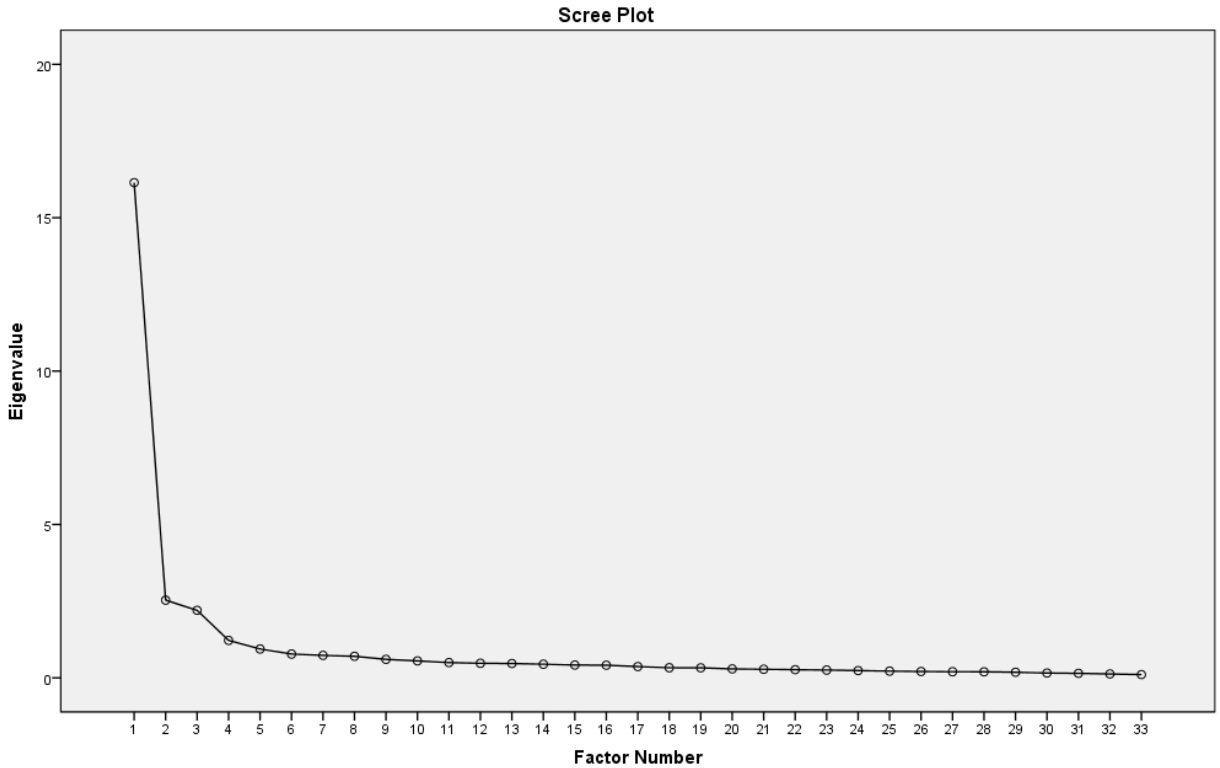


Figure 1. Scree plot from exploratory factor analysis conducted during stage eight of the SMP-KA validation process.

Table 1

Standards for Mathematical Practice

SMP Number	Title
1	Make sense of problems and persevere in solving them.
2	Reason abstractly and quantitatively.
3	Construct viable arguments and critique the reasoning of others.
4	Model with mathematics.
5	Use appropriate tools strategically.
6	Attend to precision.
7	Look for and make use of structure.
8	Look for regularity in repeated reasoning.

Note: Discussion about a specific SMP is denoted as SMP# within the chapter.

Table 2

Alignment of Stages and Actions for Validating the SMP-KA

Stage #	Description of Stage	Actions Completed During this Study	Source of Validity Evidence
1	Literature Review	Examined other SMP protocols, reviewed literature on SMPs, Knowledge, and MKT	Test Content
2	Conduct interviews with content experts and potential tool users to consider ideas on knowledge of SMPs	Conducted interviews with an expert panel consisting of K-12 math teachers, math coaches, mathematicians, mathematics teacher educators, and a mathematics curriculum coordinator and a state department of education mathematics representative.	Test Content
3	Synthesize data from literature review and interviews with content experts to discern relevant knowledge types	Employed typological analysis to generate five levels of knowledge to be assessed; Remember Factual SMP Knowledge, Understand Conceptual SMP Knowledge, Analyze Conceptual SMP Knowledge, and Creating Factual and Conceptual SMP Knowledge	Internal Structure
4	Item development	Created items for each of the five knowledge categories	—
5	Expert panel review	Submitted items of the Standards for Mathematical Practice Knowledge Assessment to expert panel	Test Content
6	Conduct interviews with potential users of tool and synthesize from these data	Conducted 1-1 and small-group cognitive interviews with K-12 math teachers, a curriculum coach, a curriculum coordinator, and a mathematics educator	Response Processes
7	Pilot testing the assessment	Collected 285 instances of SMP-KA results completed by 189 participants (approximately 50% of sample completed both a pre and post)	Internal Structure and evidence of reliability and internal consistency
8	Conducting psychometric analysis of collected assessment data	Performed exploratory factor analysis and calculated inter-rater reliability	Internal Structure and evidence of reliability and internal consistency

Table 3

SMP-KA Knowledge Alignment and Order of Administration

Phase	Knowledge Type
1	RFK
2	UCK
3	CFK & CCK
4	ACK

Note: RFK = Remembering Factual Knowledge, UCK = Understanding Conceptual Knowledge, CFK = Creating Factual Knowledge, CCK = Creating Conceptual Knowledge, ACK = Analyzing Conceptual Knowledge; Respondents must complete each phase before seeing the items in the next phase.

Table 4

Means, Standard Deviations, Pattern Coefficients, and Communalities for SMP-KA Items

			Pattern Coefficient				
Item	*M*	*SD*	RFK	UCK	CFK/CCK	ACK	h^2
Phase 1, RFK Factor (SSSC = 12.2, Cronbach's α = 0.95)							
Q1A	0.69	0.83	**0.81**	-0.04	0.02	0.06	0.71
Q1B	0.62	0.85	**0.90**	0.04	-0.12	0.05	0.79
Q1C	0.49	0.79	**0.88**	0.02	-0.13	0.10	0.82
Q1D	0.37	0.49	**0.84**	-0.10	0.01	0.01	0.66
Q1E	0.50	0.76	**0.79**	0.03	0.05	0.08	0.79
Q1F	0.32	0.48	**0.81**	-0.06	0.05	0.05	0.73
Q1G	0.50	0.83	**0.84**	0.07	0.07	-0.05	0.80
Q1H	0.31	0.70	**0.80**	0.09	0.10	-0.19	0.71
Phase 2, UCK Factor (SSSC = 12.3, Cronbach's α = 0.94)							
Q2	0.60	0.90	-0.04	**0.66**	0.11	0.11	0.67
Q3	0.36	0.73	0.12	**0.68**	0.02	0.15	0.81
Q4	0.47	0.77	0.04	**0.68**	-0.04	0.07	0.61
Q5	0.27	0.65	0.12	**0.88**	0.11	-0.19	0.79
Q6	0.36	0.64	-0.08	**0.73**	-0.09	0.12	0.54
Q7	0.34	0.69	-0.14	**0.82**	0.05	0.11	0.74
Q8	0.16	0.51	0.06	**0.81**	-0.10	0.00	0.71
Q9	0.19	0.57	0.01	**1.02**[a]	-0.04	-0.22	0.73
Phase 3, ACK Factor (SSSC = 13.1, Cronbach's α = 0.94)							
Q10	0.44	0.73	-0.10	0.10	0.07	**0.76**	0.70
Q11	0.40	0.81	0.10	-0.13	0.00	**0.76**	0.62
Q12	0.42	0.81	0.03	-0.03	-0.06	**0.79**	0.64
Q13	0.58	0.80	-0.03	0.04	0.19	**0.68**	0.70
Q14	0.39	0.74	0.11	0.10	-0.05	**0.64**	0.60
Q15	0.51	0.81	0.03	-0.01	0.13	**0.76**	0.74
Q16	0.43	0.76	0.06	0.22	0.08	**0.57**	0.74
Q17	0.40	0.78	0.11	0.13	0.26	**0.35**	0.62
Q18	0.40	0.82	0.13	0.29	0.14	**0.36**	0.71
Phase 4, CFK/CCK Factor (SSSC = 10.7, Cronbach's α = 0.87)							
Q19	0.65	0.80	0.05	0.02	**0.67**	0.01	0.52
Q20	0.33	0.60	0.04	0.03	**0.47**	0.09	0.42
Q21	0.79	0.86	-0.01	0.02	**0.84**	-0.06	0.63
Q22	0.29	0.61	-0.17	0.13	**0.44**	0.13	0.39
Q23	0.57	0.81	0.02	0.12	**0.64**	-0.12	0.46
Q24	0.43	0.68	0.13	-0.21	**0.55**	0.13	0.47
Q25	0.87	0.88	-0.10	-0.14	**0.99**	-0.01	0.66
Q26	0.40	0.71	0.17	0.07	**0.54**	-0.04	0.49

Note: Bold indicates highest pattern coefficient. h^2 = communality, SSSC = sum of squared structure coefficients.
[a] In an EFA with an oblique rotated solution, pattern coefficients are analogous to beta weights in multiple regression analyses and can fall outside the +1 to -1 range, unlike correlation coefficients.

Table 5

Factor Correlations

Factor	1	2	3	4
1. RFK	—			
2. UCK	.64	—		
3. CFK/CCK	.55	.58	—	
4. ACK	.68	.71	.73	—

Table 6

Validity Argument for the Interpretation of the SMP-KA as a Measure of Inservice Teacher's Complex Knowledge of the SMPs

Source of Validity Evidence	**Claim**	**Evidence**
Test Content	• The knowledge of the SMPs required for effective mathematics instruction is complex and multidimensional	• Confirmed by systematic review of literature before development • Confirmed through interviews with practitioners and experts before development
	• The SMP-KA adequately measures the complexity and multidimensionality of SMP knowledge	• Confirmed through expert review of the SMP-KA before piloting
Response Processes	• The SMP-KA's phase structure engages respondents in the appropriate response processes to measure the intended knowledge domain (i.e., the intended knowledge dimension and the intended cognitive process dimension)	• Confirmed through cognitive interviews with a representative sample of respondents
Internal Structure	• The SMP-KA measures the intended knowledge domain independently of the other knowledge domains	• Confirmed by exploratory factor analysis results
Relations to Other Variables	--	Currently being collected
Related Consequences	--	Currently being collected

Note: Table developed based on the recommendations of Ferrara, S. (2007). Our field needs a framework to guide development of validity research agendas and identification of validity research questions and threats to validity. *Measurement: Interdisciplinary Research and Perspectives*, *5*(2-3), 156-164. doi:10.1080/15366360701487500. Sources of validity evidence are derived from the *Standards for Educational and Psychological Testing* (AERA, APA, & NCME, 2014).