# Seeing Spots and Developing Multiplicative Sense Making

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*Cans on a grocery store shelf and Hirst's* Capric Acid Amide *can illustrate dot arrays, thus helping students understand the distributive property, partial products, and the standard algorithm for multiplication.*

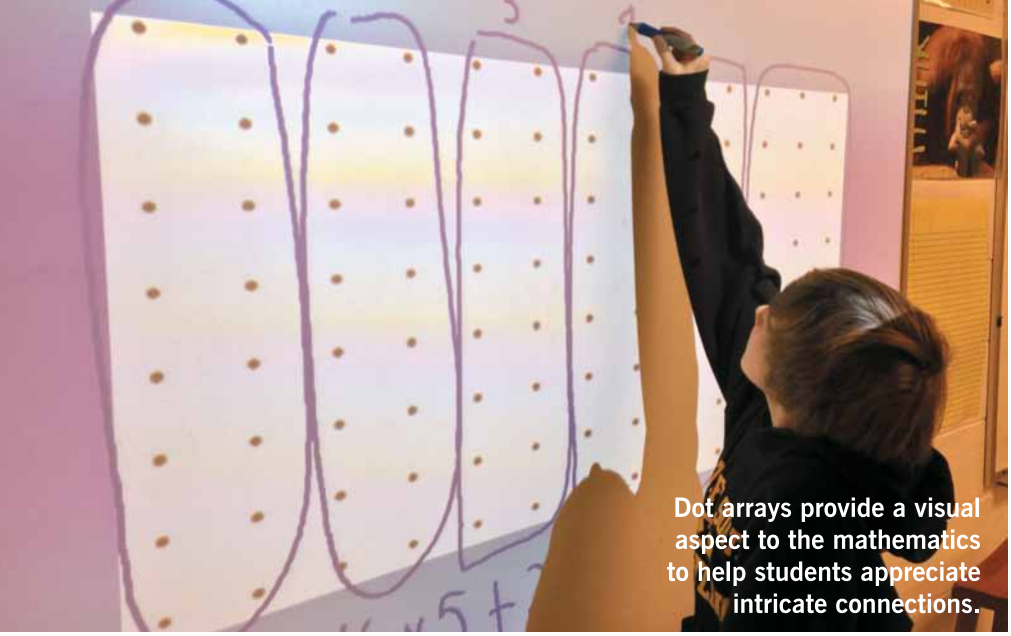
In Damien Hirst's *Capric Acid Amide* (2003), the viewer sees a lot of dots. This piece of art contains 11 rows and 14 columns of colored dots. One may ask, "What would be the most efficient way to count them all?" Is it best to think of these dots as 11 groups of 14 and add 14 eleven times? Or are there other, more efficient ways to count the dots? When students are asked to represent efficient multiplication methods and present multiple strategies, including partial products and the standard algorithm, dot arrays similar to *Capric Acid Amide* can be used to engage students visually.

ART-SIBERIA/THINKSTOCK

Dot arrays provide opportunities for students to notice structures like commutativity and distributivity, giving these properties an image that can be manipulated and explored. These images also connect to ways that we organize discrete objects in everyday life.

It is likely that you have encountered many rows and columns at your local grocery store (see **fig. 1**). Because store managers like to keep their shelves looking fully stocked, items are often layered five deep. If the cans on the shelves in the photograph are currently only three layers deep, how many cans will be needed to ensure that the shelves are fully stocked? What properties and strategies involving multiplication could be used to efficiently find an answer? We will describe how we engage students in making sense of these multiplicative situations by using dot arrays.

As middle school teachers, we have noticed that many students begin sixth grade with a limited understanding of multiplication and its properties. That is not to say that they do not know how to correctly perform two-digit and three-digit multiplication using the standard algorithm. Rather, they lack a sense of flexibility, efficiency, and accuracy (Russell 2000). When given a word problem or a realistic task, we often hear students ask, "Do I multiply or divide?" Instead of persevering on their own, they rely on us, their teachers, to provide the appropriate arithmetic procedure. In some instances, they will simply multiply two numbers without justifying whether or not the answer makes sense in the context of the problem. Students often make errors in their computation that are compounded by a lack of understanding



Dot arrays provide a visual aspect to the mathematics to help students appreciate intricate connections.

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about the process they are using. As a result, we have found it necessary to re-engage students in multiplicative thinking. By doing so, we hope to deepen their knowledge of multiplication so that they can persist when encountering multiplication in new contexts throughout middle school and high school.

## MULTIPLICATIVE SENSE MAKING

We teach with awareness of where students are in their current understanding of multiplication. At the same time, we look ahead to the mathematical learning that will connect to those understandings. The Common Core State Standards for Mathematics (CCSSM) contains numerous connections that will require fluency with multiplication beyond a memorized procedure (CCSSI 2010). As early as the sixth grade, students will need to expand their multiplicative thinking to the following:

1. Ratio and proportion (6.RP, p. 42)
2. Number system knowledge, including factoring and integer operations (6.NS, pp. 42–43)
3. Algebraic expressions and equations, in which students write expressions, identify equivalencies, and evaluate multiplicative expressions of real-world problems using algebraic notation (6.EE, pp. 43–44)
4. Geometry, in which students connect multiplicative ideas to areas of rectangles, triangles, special quadrilaterals, and volumes of prisms (6.G, pp. 44–45)

Once the full implementation of CCSSM occurs, we are hopeful that students will enter middle school with multiplicative fluency. That is to say, students' thinking about multiplication will be flexible, efficient, and precise. Such fluency can open learning spaces through which students can deepen their mathematical knowledge while making sense of CCSSM's middle school domains.

The way that CCSSM is organized in the lower grades allows this development to occur as students meet each grade's fluency mark through problem solving and modeling new mathematical ideas, early tenets of the Standards for Mathematical Practice. However, those of us working in middle schools should prepare to receive students who have not been exposed to the same set of coherent standards during their mathematics learning. In the next few years, students will most likely have some instruction implemented from CCSSM and from previous state and local standards. For these reasons, we plan to assess students' multiplicative sense making and re-engage them using activities like dot arrays to develop opportunities for future connections.

**Fig. 1** A display of cans on a grocery store shelf is a real-life model of an array representation.

## DOT ARRAYS

Our experience with individualized education plans has led us to develop dot array tasks that we have found beneficial for all students. These tasks allow students to reconnect and redevelop a sense for multiplicative efficiency and understanding, particularly with large numbers. They quickly come to realize that our expectations go beyond a memorized procedure and instead involve reasoning and justification. We begin this process with the dot array model because it allows for re-entry into multiplication by all students and promotes opportunities for creative thinking and various representations.

The structure of a dot array is particularly powerful because with each array, we are able to choose two particular grouping arrangements. We can then observe how students use (or do not use) the groups that are given. Furthermore, many students are somewhat familiar with arrays and their connection to multiplication, yet few have ever been given tasks with arrays larger than $6 \times 6$.

As the unit progresses, the sophistication of the tasks increases. Students also discover more efficient

**Fig. 2** This multiplicative re-engagement flow typifies a stream of tasks and discussions derived from the work of students.

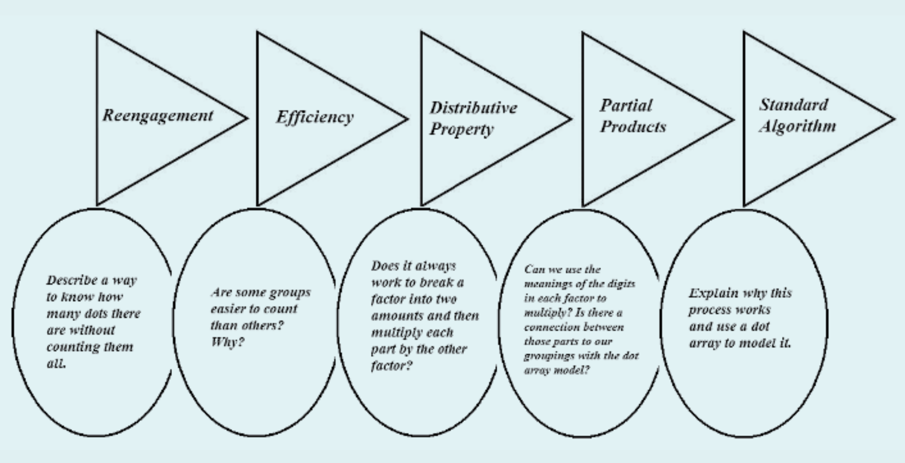
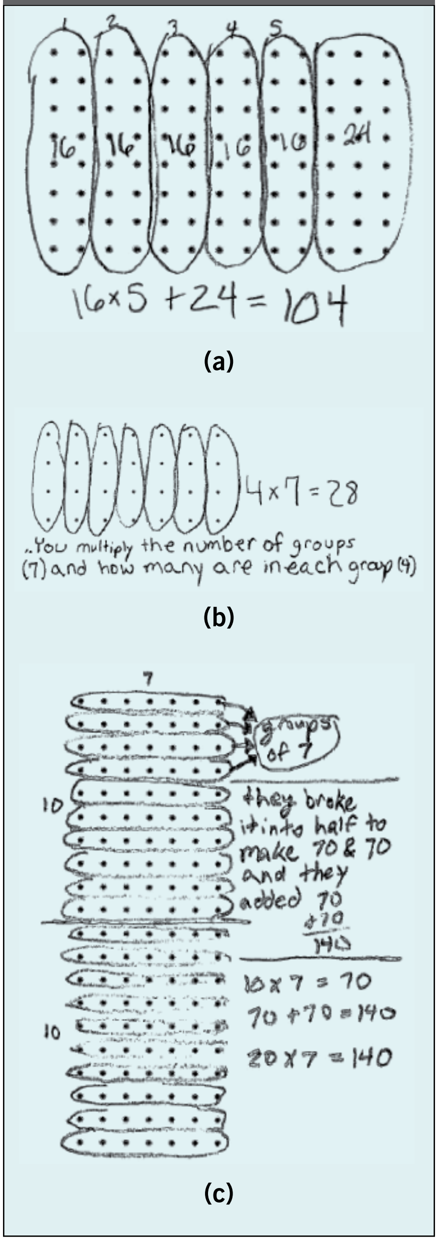



**Fig. 3** Students approach in different ways the task of finding the number of dots without counting.




strategies with the array model and connect these strategies with more abstract forms of representation. The classroom discourse ebbs and flows among exploring efficiency and representation, analyzing the thinking of others, and explaining the meaning of multiplication. **Figure 2** shows a typical stream of tasks and discussions that derive from students' work.

## RE-ENGAGEMENT INTO MULTIPLICATION

To begin, we give students dot arrays and ask them to describe some ways to ascertain the number of dots without counting them all. Students may make any decision that feels natural for them. We establish the expectation that no one can read their minds, so they must explain their strategy using drawings, words, or symbols. The openness of the task allows us to watch and consider students' thinking, highlight the differences in approaches through presentation of students' solutions, and orchestrate productive mathematics discourse (Smith and Stein 2011) about multiplication and its representations.

**Figure 3** shows examples of dot array work. As students create and share their ideas, we are looking for examples that can both expand and deepen their mathematical views. In **figure 3a**, the student used an $8 \times 13$ array and explained, "Five groups of sixteen is easy, that's 80, so I just added the 24 and made 104." The idea that we can work out multiplication problems using a combination of addition with smaller multiplications is significant, as it moves beyond the idea of $8 \times 13$ as meaning only 8 groups of 13.

A solution like that in **figure 3a** can act as a catapult, causing students to think about other meanings for multiplication. After discussing solutions that use addition with smaller examples of multiplications, students gain flexibility that they can later transfer to the more abstract symbolic meaning involved in explaining algorithms.

The example in **figure 3b** can also elicit discussions about the need for precision. This student's written expression matches the drawing by making 7 groups of 4. However, the number sentence was written as $4 \times 7$, implying 4 groups of 7. Several students typically represent their written multiplications in a way that does not follow convention, but they are given the opportunity to explain the meaning and demonstrate visually how the expression connects with their dot array groupings. These cases provide fresh opportunities to engage students in a discussion about the meaning and conventional representation of multiplicative expressions.

An important question that we raise for consideration is "How can dot arrays be used to give a justification for the commutative property?" We allow time for students to discuss the issue and consider why it might be advantageous to think of $4 \times 7$ and $7 \times 4$ as different, even though the

**Fig. 4** To find the product of 22 and 16, one student grouped the dots by 10s, trying to cluster 100 dots when possible.

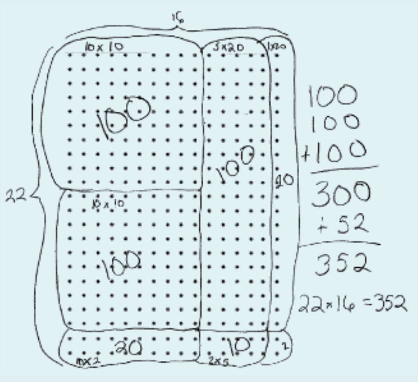
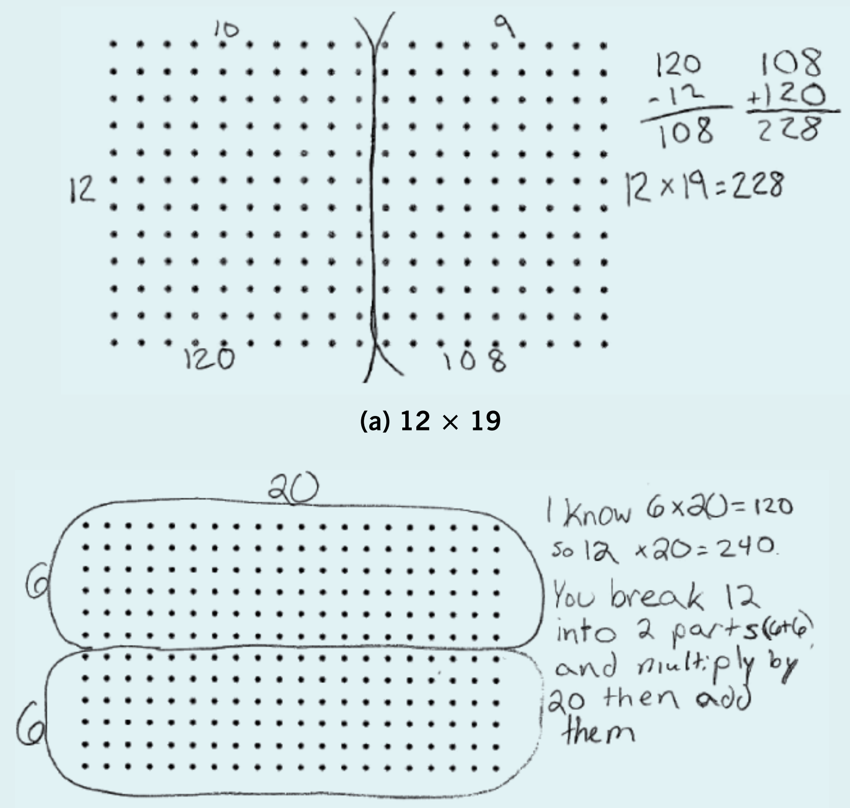



**Fig. 5** The work of these students moved them toward thinking about the distributive property.




(a) 12 × 19

(b) 12 × 20

product is the same by the commutative property. Students demonstrate that each $M \times N$ array can be turned 90 degrees to make an $N \times M$ array without changing the original.

Two other important aspects that are generated by these dot array re-engagements involve understanding and critiquing the reasoning of others and moving toward efficiency. We allow students to work in pairs and bounce ideas off each other. When students believe that they have a finished solution, their ideas are tested by another pair, to see if they can discover the ideas and whether or not the drawings, words, and symbols make sense. The work in **figure 3c** contains the text "They broke it into half to make 70 & 70. . . ." These words were added later by another pair of students who were checking if enough information was provided to deduce a strategy. Students can negotiate with one another about what needs to be done to accurately state the intended strategy.

Another important aspect that we want to elevate through discourse is the careful choice of grouping that can produce an easier solution. **Figure 3c** was shared to promote this discussion. The idea of breaking 20 into 2 tens made it easier for students to find the answer of $20 \times 7$. The class decided that this strategy was more efficient than others. We highlight these examples as a natural transition to our discussions and expectations of increasing efficiency.

## EFFICIENCY

Students are given more dot arrays with the expectation of finding groups that relate to the base ten number system. They may organize these groups in whatever way makes the most sense to them, often choosing to find multiple groups of 10. The sample work of **figure 4** shows that this student chose to make multiple groups of 10 or 20 to form larger groups of 100. Throughout their presentations and discussions, we encourage students to think about ways that they can improve their own efficiency.

In subsequent conversations, students often referred to others' ideas, such as "Joe's method," to indicate whose thoughts they were using to justify their new adaption. Although the approach for obtaining $22 \times 16$ used in **figure 4** is more advanced in efficiency than others', connections can be made that will produce improvement. One such connection involves the move by some students to break up the main factors of the array to make more readily known multiplications over addition. When these cases occur, we discuss these strategies (see **fig. 5**) and their connection to the distributive property.

## DISTRIBUTIVE PROPERTY

Dot arrays can provide a clear picture of the distributive property because students can see how one factor is broken apart in ways that make multiplication by the other factor easier. In students' attempts to demonstrate a more efficient method using the dot array model, we invariably see work that exemplifies the distributive property. We look for the sort of student work shown in **figure 5** to highlight—

- a concise representation of ideas;
- a decomposition of factors within any multiplication problem to make it easier to solve; and
- a consideration of whether or not both factors can be decomposed.

After being given time to think, students realize that the work in **figure 5a** and **5b**, respectively, can be written concisely as the following:

- $$\begin{aligned} 12 \times 19 &= 12 \times (10 + 9) \\ &= 120 + 108 \\ &= 228 \end{aligned}$$
- $$\begin{aligned} 12 \times 20 &= (6 + 6) \times 20 \\ &= 120 + 120 \\ &= 240 \end{aligned}$$

We ask students to work through other dot arrays and try this strategy because it opens a space for students to look for and make use of structure. It provides important experiences for their future consideration of $a(b + c) = ab + ac$. Looking beyond the trivial form of the distributive property, we ask students to consider what it would mean if we decomposed both factors into numbers of our choice and then multiplied. Often, students bring up this idea themselves as they notice nice groups that they have formed in their dot arrays. From these ponderings arise connections to partial products.

## PARTIAL PRODUCTS

Some students' ideas gravitate naturally toward modeling the partial products algorithm (see **fig. 6**) or toward dividing the dots into sections and finding the number of dots in each section. During our challenge to students to be more efficient, some of them discovered this strategy and decomposed both factors, although not always choosing the same decomposition. This situation provided the context for us to ask, "Can the multiplications in the partial products algorithm be represented by a dot array?" Students were initially allowed to explore this question with their choice of any size dot array. Many students worked through several cases and discussed their findings with others.

After students' ideas were shared, we asked them to demonstrate the partial products algorithm using a 39 × 14 array. The student work in **figure 6** illustrates how each of the partial multiplications is geometrically parsed. We find that students benefit the most when we give them the task

**Fig. 6** The partial products algorithm was modeled in this dot array.

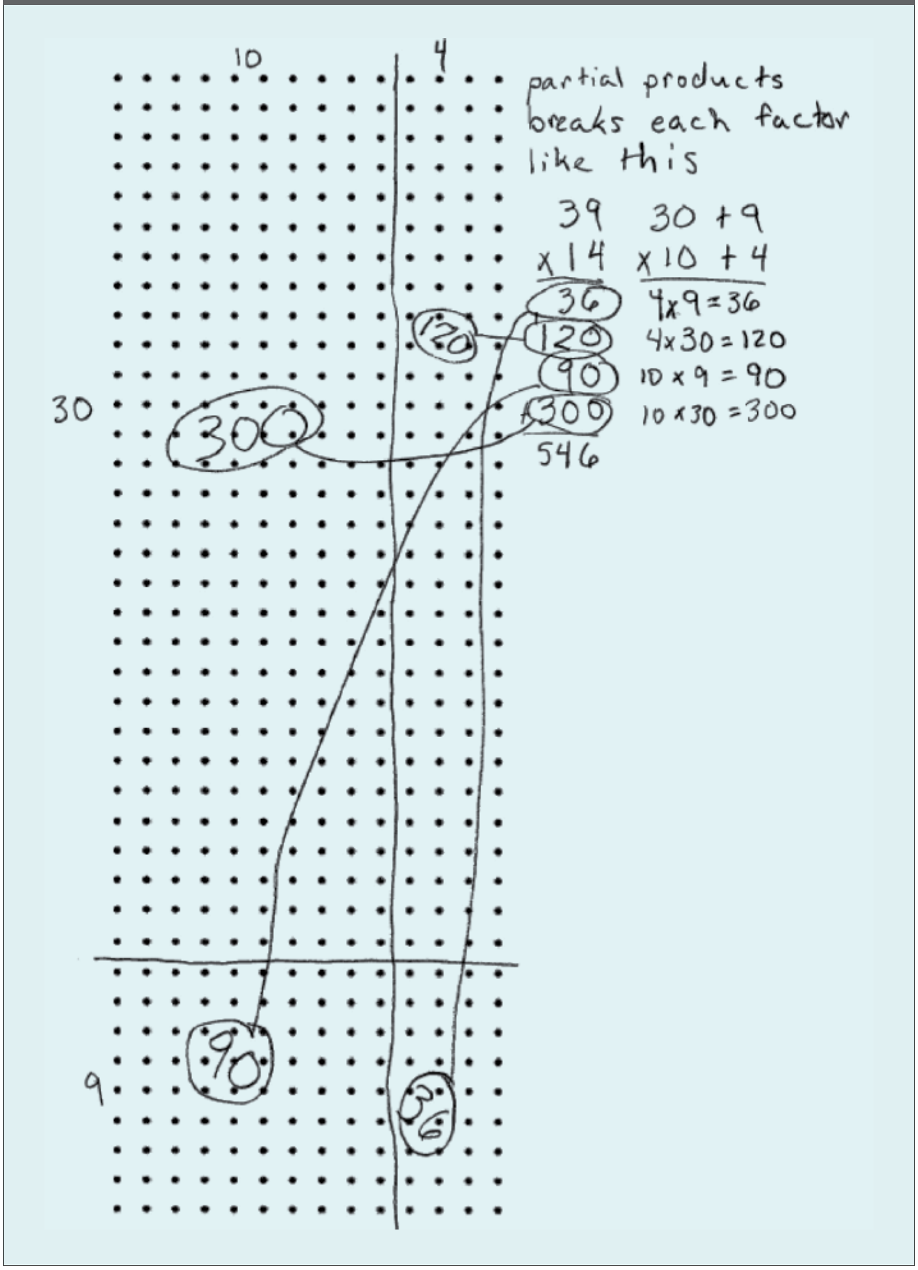
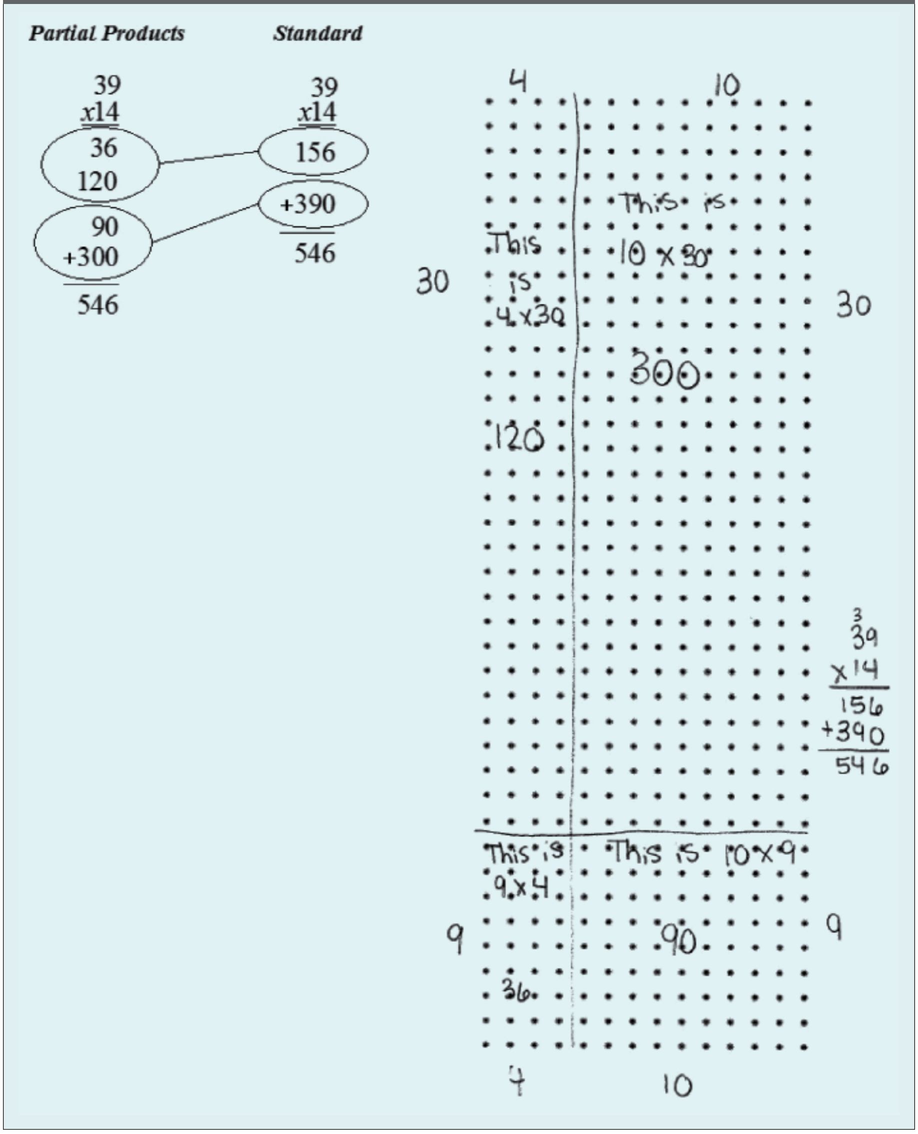


**Fig. 7** Through their experiences with dot arrays, students can see that the standard algorithm for multiplication is consistent with their work in partial products.




of connecting the place value meaning of the digits and the multiplicative process to the dot array model.

We often hear students make clarifying remarks about situations that we as teachers take for granted. During one class, a student exclaimed, "Oh! So that's why we add all of these numbers up after we multiply the pieces! I always thought that was weird, but yeah, they are just pieces of the whole." The dot arrays provide a visual model that can help students better understand the arithmetic processes, including the standard algorithm for multiplication.

## STANDARD ALGORITHM

We conclude the re-engagement with multiplication by asking pairs of students to explain the process of the standard algorithm for multiplication using the dot array model. Students quickly see that the smaller products of the standard algorithm match those of the partial products algorithm. They also notice that the standard algorithm takes less writing because its representation allows two numbers to be summed in one step, as shown in **figure 7**.

We encourage students to be precise in their explanations and to ask questions of one another to account for all the details in the standard algorithm. "Why do we write the 3 from the 30 up there?" and "Why were we told to put a zero in the one's place on the second row?" are two of our favorite questions. After much discussion and presentation with their peers, each group writes an explanation and presents it to the class. The students actively seek to answer the questions that arise during discussion. We have found that students give richer explanations of the standard algorithm for multiplication when they have experiences with and use the dot array model.

Many students need the visual aspects to appreciate the intricacies and connections involved in mathematics. The dot array model allows students the freedom to explore mathematically appropriate representations of arrays and learn how to make more efficient decisions. Through these multiplication experiences, students can entertain mathematical questions like "In trying to find the product of $11 \times 14$, is it best to think of it as 11 groups of 14 and add 14 eleven times, or is there another more efficient way?"

## IDEAS ON THE HORIZON

These activities prepare students for making many of the mathematical connections they will encounter in their immediate future. For example, when we move into algebraically representing the distributive property, we invite students to think back to the dot arrays by asking, "How can we represent $12(x + 7)$?" We have also found that students who have significant experiences with dot arrays are much more adept at transitioning to similar models comprising more sophisticated forms of multiplication. For example, we have seen students use algebra tiles to think through multiplication of binomials. Student strategies also tend to transfer from the discrete nature of the dot array to the continuous nature of solving area problems. As they solve area problems, we have noticed an increase in their self-initiative in drawing auxiliary lines to break up areas into more manageable pieces.

We have described many ways that dot arrays can be used to deepen students' understanding of multiplication. Our school system has a transient population of students who are ESL and impoverished. Many of the older students who come to us do so with vast deficits in their knowledge. These same tasks were used by teachers at the high school level to help students make sense of multiplication and apply this knowledge in their algebra courses. We encourage middle school and high school teachers to present the dot array model to their students who struggle to make sense of multiplicative ideas.

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